JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 155, 140–155 (1991)

# A Strict Dual Representation for $L_1(\mu, X)$

MARK E. GRUENWALD

*Department of Mathematics, University of Evansville, Evansville, Indiana 47722*

AND

ROBERT F. WHEELER

*Department of Mathematical Sciences, Northern Illinois University, DeKalb, Illinois 60115*

*Submitted by C. Foias*

Received August 18, 1989

Let $(\Omega, \Sigma, \mu)$ be a complete finite measure space, let $X$ be a Banach space, and let $S$ be the Stone space of the measure algebra $\Sigma/\mu^{-1}(0)$. We define strict topologies $\beta_1$ and $\beta_2$ on $C(S, X_\tau^*)$, the space of continuous functions from $S$ into $X^*$ with the Mackey topology. Using a differentiation criterion for vector measures involving a strong form of bounded variation, we show that the dual spaces $(C(S, X_\tau^*), \beta_1)'$ and $(C(S, X_\tau^*), \beta_2)'$ can be represented as $L_1(\mu, X)$, the space of Bochner integrable $X$-valued functions on $\Omega$. Finally, we characterize the subsets of $L_1(\mu, X)$ which are equicontinuous with respect to the $\beta_1$ and $\beta_2$ topologies. © 1991 Academic Press, Inc.

## 1. INTRODUCTION

The theory of strict topologies grew out of the desire to express certain spaces of measures as duals of corresponding spaces of continuous functions. Our desire here is to consider those measures which take their values in a Banach space $X$ and which can be identified—via integration—with the space $L_1(\mu, X)$ of Bochner integrable $X$-valued functions. Basic results on the strict topology for spaces of vector-valued continuous functions can be found in [2, 10]. Our main reference for the theory of vector measures and Bochner integrable functions is the book of Diestel and Uhl [3].

A major goal of this paper is to use a differentiability criterion (Lemma 2.2) to represent $L_1(\mu, X)$ as the dual of an appropriate space of continuous functions with an appropriate strict topology. To that end, we consider a space of continuous functions which can be viewed as a sub-



0022-247X/91 $3.00

space of the dual of $L_1(\mu, X)$. Let $S$ be the Stone space of the measure algebra $\Sigma/\mu^{-1}(0)$ and denote by $C(S, X^*_{w^*})$ the space of continuous functions from $S$ into $X^*$ with the weak* topology $\sigma(X^*, X)$. Using the Stonian transform techniques of D. Sentilles [8], one can show that $C(S, X^*_{w^*})$ is isomorphic to $L_1(\mu, X)^* = L_\infty(\mu, X^*, X)$, the space of ($w^*$-equivalence classes of) bounded, $X^*$-valued, $w^*$-measurable functions on $\Omega$. In Section 3, we introduce the strict topologies $\beta_1$ and $\beta_2$ on $C(S, X^*_\tau)$, a subspace of $C(S, X^*_{w^*})$. With the help of the differentiability criterion, Section 4 reveals that $L_1(\mu, X)$ can be represented as either $(C(S, X^*_\tau), \beta_1)'$ or $(C(S, X^*_\tau), \beta_2)'$. The definitions of the topologies $\beta_1$ and $\beta_2$, as well as the techniques used to represent $L_1(\mu, X)$ as the strict dual of $C(S, X^*_\tau)$, were originally developed in [11], and are studied in detail in [6].

In Section 5, we consider the subsets of $L_1(\mu, X)$ which are equicontinuous with respect to the strict topologies $\beta_1$ and $\beta_2$. In addition to their importance to the Mackey problem, these subsets turn out to have some interesting connections to the subsets of $L_1(\mu, X)$ which are relatively compact in the weak topology $\sigma(L_1(\mu, X), L_1(\mu, X)^*)$. These issues will be examined in a future paper.

## 2. A Differentiation Criterion

Let $(\Omega, \Sigma, \mu)$ be a finite measure space, let $X$ be a Banach space, and let $F\colon \Sigma \to X$ be a vector measure. The *variation* of $F$ is the extended non-negative set function $|F|$ defined for each $A \in \Sigma$ by

$$|F|\,(A) = \sup\left\{\sum_{i=1}^{m} \|F(A_i)\| : \{A_i\}_{i=1}^{m} \text{ is a } \Sigma\text{-partition of } A\right\}.$$

$F$ is said to be of *bounded variation* if $|F|\,(\Omega) < \infty$. A stronger notion of vector measure variation can be introduced which is closely related to the $p$-variation of a vector measure considered by Wells [10]. Given a subset $L$ of $X$, the *$L$-variation* of $F$ is the extended non-negative set function $|F|_L$ defined for each $A \in \Sigma$ by

$$|F|_L\,(A) = \sup\left\{\sum_{i=1}^{m} |\langle F(A_i), x_i^*\rangle| : \right.$$
$$\left. \{A_i\}_{i=1}^{m} \text{ is a } \Sigma\text{-partition of } \Omega,\ \{x_i^*\}_{i=1}^{m} \subset L^o\right\}.$$

$F$ is said to be of *bounded $L$-variation* if $|F|_L\,(\Omega) < \infty$. The following properties involving bounded $L$-variation will be useful later.

THEOREM 2.1. *Let $F: \Sigma \to X$ be a vector measure which is bounded in $L$-variation for some subset $L$ of $X$.*

(1) *If $|F|_L(\Omega) = \lambda$, then $\{F(A): A \in \Sigma\} \subset \lambda L^{oo}$.*

(2) *If $M$ is another subset of $X$ with $L \subset \gamma M$ for some $\gamma > 0$, then $|F|_M(\Omega) \leqslant \gamma |F|_L(\Omega)$ and so $F$ is of bounded $M$-variation.*

(3) *If $L$ is bounded, then $F$ is of bounded variation in the usual sense. If $L = B_X$, then in fact $|F|_L(\Omega) = |F|(\Omega)$ and so $F$ is of bounded $L$-variation iff it is of bounded variation.*

(4) *For each $A \in \Sigma$, $|F|(A) = 0$ only if $|F|_L(A) = 0$. If $L$ is bounded, then the converse implication holds as well.*

(5) *If $L$ is bounded and $F$ is countably additive and $\mu$-continuous, then $|F|_L$ is a finite, countably additive, $\mu$-continuous scalar measure on $\Sigma$.*

*Proof.* (1), (2), (3), and (5) are easily verified.

(4) The "only if" direction is clear from the definitions of $|F|$ and $|F|_L$. If $L$ is bounded, then $B_{X^*} \subset \gamma L^o$ for some $\gamma > 0$. Thus, if $|F|_L(A) = 0$, then $|\langle F(A), x^* \rangle| = 0$ for all $x^* \in B_{X^*}$ and it follows that $F(A) = 0$. Note that this converse does not hold if $L$ is not bounded. Consider, for example, $X = L = \mathbf{R}$ with $(\Omega, \Sigma, \mu)$ the Lebesgue measure on $[0, 1]$. In this case, $L^o = \{0\}$ and so $|\mu|_L(A) = 0$ for all $A \in \Sigma$. ∎

For certain choices of the subset $L$, there is a close relationship between vector measures of bounded $L$-variation and differentiability. The following lemma establishes this relationship.

LEMMA 2.2 (Differentiation Lemma). *Let $F: \Sigma \to X$ be a countably additive, $\mu$-continuous vector measure of bounded variation. Then the following are equivalent:*

(a) *$F$ is differentiable (i.e., there is a function $f \in L_1(\mu, X)$ such $F(A) = \int_A f \, d\mu$ for all $A \in \Sigma$).*

(b) *There is a norm compact subset $K$ of $X$ such that $F$ is of bounded $K$-variation.*

(c) *There is weakly compact subset $L$ of $X$ such $F$ is of bounded $L$-variation.*

*Proof.* (a) $\Rightarrow$ (b) Let $f \in L_1(\mu, X)$ be such that $F(A) = \int_A f \, d\mu$ for all $A \in \Sigma$. Since $L_1(\mu, X) = L_1(\mu) \hat{\otimes} X$, there is an $\alpha = (\alpha_j)_{j=1}^{\infty} \in l_1$, a sequence $\{g_j\}_{j=1}^{\infty}$ of functions in $L_1(\mu)$ with $\|g_j\| \leqslant 1$ for all $j \in \mathbf{N}$ and $\|g_j\| \to 0$, and a sequence $\{x_j\}_{j=1}^{\infty}$ in $X$ with $\|x_j\| \leqslant 1$ for all $j \in \mathbf{N}$ and $\|x_j\| \to 0$ such that $f(\omega) = \sum_{j=1}^{\infty} \alpha_j g_j(\omega) x_j$ for all $\omega \in \Omega$ [5]. Let $K = \{x_j\}_{j=1}^{\infty} \cup \{0\}$, a norm compact subset of $X$. Then for any $\Sigma$-partition $\{A_i\}_{i=1}^m$ of $\Omega$ and any collection $\{x_i^*\}_{i=1}^m \subset K^o$,

$$\begin{aligned}
\sum_{i=1}^m |\langle F(A_i), x_i^*\rangle| &= \sum_{i=1}^m \left|\left\langle \int_{A_i} f\, d\mu, x_i^* \right\rangle\right| \\
&= \sum_{i=1}^m \left| \sum_{j=1}^\infty \alpha_j \left(\int_{A_i} g_j\, d\mu\right) \langle x_j, x_i^*\rangle \right| \\
&\leqslant \sum_{i=1}^m \sum_{j=1}^\infty |\alpha_j| \int_{A_i} |g_j|\, d\mu \\
&= \sum_{j=1}^\infty |\alpha_j| \left( \sum_{i=1}^m \int_{A_i} |g_j|\, d\mu \right) \\
&= \sum_{j=1}^\infty |\alpha_j|\, \|g_j\|_1 \\
&\leqslant \|\alpha\|.
\end{aligned}$$

Thus, $|F|_K(\Omega) \leqslant \|\alpha\| < \infty$.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (a) Using Theorem 2.1(2), one can assume without loss of generality that $L$ is absolutely convex. We apply the Rieffel criteria (see [1, p. 23, Theorem 2.2.6]) to show that $F$ is differentiable. That is, given $\varepsilon > 0$, we show that there is a set $\Omega_\varepsilon \in \Sigma$ such that $\mu(\Omega \backslash \Omega_\varepsilon) < \varepsilon$ and the average range $\mathrm{AR}(F|_{\Sigma \cap \Omega_\varepsilon})$ is relatively weakly compact in $X$. According to Theorem 2.1(5), $|F|_L$ is a finite, countably additive, $\mu$-continuous scalar measure on $\Sigma$. Thus, there is a $g \in L_1(\mu)^+$ such that $|F|_L(A) = \int_A g\, d\mu$ for all $A \in \Sigma$. Let $\varepsilon > 0$ and choose $\Omega_\varepsilon \in \Sigma$ such that $\mu(\Omega \backslash \Omega_\varepsilon) < \varepsilon$ and $g\chi_{\Omega_\varepsilon} \in L_\infty(\mu)$. Then for any $A \in \Sigma \cap \Omega_\varepsilon$,

$$|F|_L(A) \leqslant \mu(A)\, \|g\chi_{\Omega_\varepsilon}\|_\infty.$$

Theorem 2.1(1) and the bipolar theorem now show that

$$\frac{F(A)}{\mu(A)} \in \|g\chi_{\Omega_\varepsilon}\|_\infty\, L$$

for all $A \in \Sigma \cap \Omega_\varepsilon$. Thus, $\mathrm{AR}(F|_{\Sigma \cap \Omega_\varepsilon}) \subset \|g\chi_{\Omega_\varepsilon}\|_\infty\, L$ and the result follows. ∎

In light of Lemma 2.2 and the fact that the Mackey topology $\tau$ on $X^*$ is the topology of uniform convergence on weakly compact subsets of $X$, it is perhaps not unreasonable to expect that $X_\tau^*$ would have some connection to those $X$-valued vector measures which are differentiable with respect to $\mu$. This is indeed the case since strict topologies $\beta_1$ and $\beta_2$ can be defined on the space $C(S, X_\tau^*)$ in such a way that $(C(S, X_\tau^*), \beta_1)' = (C(S, X_\tau^*), \beta_2)' = L_1(\mu, X)$.

## 3. Strict Topologies on $C(S, X_\tau^*)$

Let $S$ be the Stone space of the measure algebra $\Sigma/\mu^{-1}(0)$ and let $T$ be a dense, $\sigma$-compact, locally compact subset of $S$. Denote by $(C_b(T, X_\tau^*))\, C(S, X_\tau^*)$ the space of (bounded) continuous functions from $(T)S$ into $X^*$ with the Mackey topology $\tau(X^*, X)$. Since the Mackey topology $\tau$ on $X^*$ is the locally convex topology generated by the family of semi-norms

$$P = \{p_L : L \text{ is a weakly compact subset of } X\},$$

where $p_L$ is defined for each $x^* \in X^*$ by

$$p_L(x^*) = \sup_{x \in L} |\langle x, x^* \rangle|,$$

one can consider the strict topologies $\beta_{p_L}$ on $C_b(T, X_\tau^*)$ for $p_L \in P$ as defined in the following way. For each $h \in C_0(T)$ and $p_L \in P$, define the semi-norm $\|\cdot\|_{h,L}$ on $C_b(T, X_\tau^*)$ by

$$\|g\|_{h,L} = \sup_{t \in T} p_L(h(t)\, g(t)) = \sup_{t \in T, x \in L} |h(t) \langle x, g(t) \rangle|$$

for each $g \in C_b(T, X_\tau^*)$. For fixed $p_L \in P$, denote by $\beta_{p_L}$ the locally convex topology on $C_b(T, X_\tau^*)$ generated by the family of semi-norms $\{\|\cdot\|_{h,L} : h \in C_0(T)\}$. To indicate its dependence on the subset $T$, we write $\beta_{TL} = \beta_{p_L}$. Likewise, one can consider the topology $\beta_T$ defined on $C_b(T, X_\tau^*)$ by

$$\beta_T = \varprojlim_L \beta_{TL},$$

the projective limit of the topologies $\beta_{TL}$ as $L$ runs over all weakly compact subsets of $X$. The topology $\beta_T$ is nothing more than Buck's strict topology $\beta$ on $C_b(T, X_\tau^*)$.

Since $C(S, X_\tau^*)$ can be thought of as a linear subspace of $C_b(T, X_\tau^*)$, $\beta_{TL}$ and $\beta_T$ will also denote the subspace topologies on $C(S, X_\tau^*)$. More specifically, for a weakly compact subset $L$ of $X$, the topology $\beta_{TL}$ on $C(S, X_\tau^*)$ has a defining base of semi-norms given by $\{\|\cdot\|_{h,L} : h \in C_0(T)\}$, where

$\|\cdot\|_{h,L}$ is defined for each $g \in C(S, X_\tau^*)$ as above. Similarly, the topology $\beta_T$ on $C(S, X_\tau^*)$ has a defining base of semi-norms given by

$$\{\|\cdot\|_{h,L} \colon h \in C_0(T),\ L \text{ is a weakly compact subset of } X\}.$$

The strict topology $\beta_1$ on $C(S, X_\tau^*)$ is now defined to be the inductive limit of the topologies $\beta_T$ as $T$ runs over all dense, $\sigma$-compact, locally compact subsets of $S$. Thus,

$$\beta_1 = \varinjlim_T \beta_T = \varinjlim_T (\varprojlim_L \beta_{TL})$$

and it is the finest locally convex topology on $C(S, X_\tau^*)$ which is coarser than $\beta_T$ for each $T$.

It is natural to inquire about the topology that results from reversing the order of the inductive and projective limits in the definition of $\beta_1$. To that end, fix a weakly compact subset $L$ of $X$ and define the topology $\beta_L$ on $C(S, X_\tau^*)$ by

$$\beta_L = \varinjlim_T \beta_{TL},$$

the inductive limit of the topologies $\beta_{TL}$ as $T$ runs over all dense, $\sigma$-compact, locally compact subsets of $S$. Finally, define the strict topology $\beta_2$ on $C(S, X_\tau^*)$ to be the projective limit of the topologies $\beta_L$ as $L$ runs over all weakly compact subsets of $X$. Thus,

$$\beta_2 = \varprojlim_L \beta_L = \varprojlim_L (\varinjlim_T \beta_{TL})$$

and it is the coarsest locally convex topology on $C(S, X_\tau^*)$ which is finer than $\beta_L$ for each $L$.

The following theorem records some interesting facts about the relationships among the various strict topologies on $C(S, X_\tau^*)$.

THEOREM 3.1. (1) *$\beta_L \leqslant \beta_{TL} \leqslant \beta_T$ and $\beta_L \leqslant \beta_2 \leqslant \beta_1 \leqslant \beta_T$ for any dense, $\sigma$-compact, locally compact subset $T$ of $S$ and any weakly compact subset $L$ of $X$.*

(2) *If $(\Omega, \Sigma, \mu)$ is purely atomic, then there is a dense, $\sigma$-compact, locally compact subset $T_0$ of $S$ such that $\beta_2 = \beta_{T_0} = \beta_1$.*

(3) *If $X$ is reflexive, then $\beta_2 = \beta_{L_0} = \beta_1$, where $L_0 = B_X$.*

(4) *If $X$ is reflexive and $(\Omega, \Sigma, \mu)$ is purely atomic, then there is a dense, $\sigma$-compact, locally compact subset $T_0$ of $S$ such that $\beta_2 = \beta_{T_0 L_0} = \beta_1$, where $L_0 = B_X$.*

*Proof.* (1) The relationships $\beta_L \leqslant \beta_{TL} \leqslant \beta_T$, $\beta_L \leqslant \beta_2$, and $\beta_1 \leqslant \beta_T$ follow directly from the inductive and projective limit definitions of these topologies. We show that $\beta_2 \leqslant \beta_1$. Let $U$ be a $\beta_2$-neighborhood of zero in $C(S, X_\tau^*)$. Then $U$ is a $\beta_L$-neighborhood of zero for some weakly compact subset $L$ of $X$ and so $U$ is a $\beta_{TL}$-neighborhood of zero for each dense, $\sigma$-compact, locally compact subset $T$ of $S$. Thus, $U$ is a $\beta_T$-neighborhood of zero for each such $T$ and hence $U$ is a $\beta_1$-neighborhood of zero.

(2) Since $(\Omega, \Sigma, \mu)$ is purely atomic, it has a maximal pairwise disjoint family of atoms. Moreover, this family is necessarily countable since the measure space is finite. Thus, the measure algebra corresponding to $(\Omega, \Sigma, \mu)$ is isomorphic to that of $(\mathbf{N}, \mathscr{P}(\mathbf{N}), \sum_{n=1}^{\infty} c_n \delta_n)$ where $\sum_{n=1}^{\infty} |c_n| < \infty$. Since $S = \mathrm{St}(\mathbf{N}) = \beta\mathbf{N}$, $T_0 = \mathbf{N}$ is the minimal dense, $\sigma$-compact, locally compact subset of $S$. It follows that

$$\begin{aligned}\beta_2 &= \varinjlim_L \, (\varinjlim_T \beta_{TL}) = \varinjlim_L \beta_{T_0}L \\ &= \beta_{T_0} \\ &= \varinjlim_T \beta_T = \beta_1.\end{aligned}$$

(3) Since $X$ is reflexive, $L_0 = B_X$ is a weakly compact subset of $X$ which generates the Mackey topology on $X^*$.

(4) Combine (2) and (3). ∎

Theorem 3.1 raises several questions. First, if $(\Omega, \Sigma, \mu)$ is not purely atomic, is it still possible to find a single dense, $\sigma$-compact, locally compact subset $T_0$ of $S$ for which $\beta_{T_0} = \beta_1$? The following example shows that it need not be possible, even if $X$ is reflexive.

EXAMPLE 3.2. Let $(\Omega, \Sigma, \mu)$ be a measure space which is not purely atomic, let $X$ be reflexive, and let $T$ be a dense, $\sigma$-compact, locally compact subset of $S$. Let $\hat{\mu}$ be the regular Borel measure on $S$ corresponding to $\mu$. We show that there is a dense, $\sigma$-compact, locally compact subset $\tilde{T}$ of $S$ such that $\tilde{T} \subset T$ and $\beta_{\tilde{T}}$ is strictly coarser than $\beta_T$. Thus, $\beta_1$ is strictly coarser than $\beta_T$. Write $T = \bigcup_{n=1}^{\infty} T_n$, where the $T_n$ are pairwise disjoint compact-open subsets of $S$. Since $(\Omega, \Sigma, \mu)$ is not purely atomic, there is a $k \in \mathbf{N}$ such that $T_k$ properly contains the union of an infinite sequence of pairwise disjoint compact-open sets, $\{\tilde{T}_n\}_{n=1}^{\infty}$ say. Without loss of generality, one can assume that $\bigcup_{n=1}^{\infty} \tilde{T}_n$ is dense in $T_k$. Define $\tilde{T} = (T \setminus T_k) \cup (\bigcup_{n=1}^{\infty} \tilde{T}_n)$ and note that $\tilde{T} \subset T$. Define $h$: $T \to \mathbf{R}$ by $h = \sum_{n=1}^{\infty} \hat{\mu}(T_n) \chi_{T_n}$ and let $L = B_X$. Then $h \in C_0(T)$ and the set

$$V = \{ g \in C(S, X_\tau^*) \colon \sup_{t \in T, x \in L} |\langle x, h(t)\, g(t) \rangle| \leqslant 1 \}$$

is a $\beta_{TL}$-neighborhood of zero in $C(S, X_\tau^*)$. We show that $V$ is not a $\beta_{\tilde{T}L}$-neighborhood of zero. Let $\tilde{h} \in C_0(\tilde{T})$ be such that $\tilde{h}(t) \neq 0$ for all $t \in \tilde{T}$ and choose $m \in \mathbb{N}$ so that $|\tilde{h}(t)| < \hat{\mu}(T_k)$ for all $t \in \tilde{T}_m$. Choose $x^* \in X^*$ such that $\|x^*\| = 1$ and define $\tilde{g}\colon S \to X^*$ by

$$\tilde{g} = \frac{1}{\tilde{h}} x^* \chi_{\tilde{T}_m}.$$

Then $\tilde{g} \in C(S, X_\tau^*)$. Moreover, if $t \in \tilde{T}$ and $x \in L$, then

$$|\langle x, \tilde{h}(t)\, \tilde{g}(t)\rangle| \leqslant \chi_{\tilde{T}_m}(t).$$

So $\tilde{g} \in \{g \in C(S, X_\tau^*)\colon \|g\|_{\tilde{h}, L} \leqslant 1\}$. But if $t \in \tilde{T}_m$, then

$$\sup_{x \in L} |\langle x, h(t)\, \tilde{g}(t)\rangle| = \frac{\hat{\mu}(T_k)}{|\tilde{h}(t)|} \|x^*\| > 1.$$

So $\tilde{g} \notin V$ and it follows that there is no basic $\beta_{\tilde{T}}$-neighborhood of zero in $C(S, X_\tau^*)$ which is contained in $V$. Thus, $V$ is not a $\beta_{\tilde{T}}$-neighborhood of zero in $C(S, X_\tau^*)$. ∎

The second question raised by Theorem 3.1 is whether or not there are other classes of Banach spaces for which $\beta_L = \beta_2$ for some weakly compact subsét $L$. The likely candidates are those Banach spaces which are strongly weakly compactly generated (SWCG). This term, introduced by G. Schlüchtermann and R. Wheeler [9], refers to those Banach spaces $X$ for which there is a weakly compact subset $L$ of $X$ with the property that whenever $M$ is a weakly compact subset of $X$ and $\varepsilon > 0$, there is an $n \in \mathbb{N}$ with $\bar{M} \subset nL + \varepsilon B_X$. Reflexive spaces and separable Schur spaces are both examples of SWCG spaces. The second question can now be phrased in the following way: If $X$ is an SWCG space with $L$ a strongly generating weakly compact subset of $X$, is it true that $\beta_L = \beta_2$? The next example indicates that this is not true in general, even for purely atomic measure spaces.

EXAMPLE 3.3. Let $X = l_1$ (an SWCG space) and let $L$ be a weakly compact subset of $X$. We show that there is a weakly compact subset $M$ of $X$ such that $L \subset M$ and $\beta_{TL}$ is strictly coarser than $\beta_{TM}$ for each dense, $\sigma$-compact, locally compact subset $T$ of $S$. Thus, $\beta_{TL}$ is strictly coarser than $\beta_T$ for each such $T$ and, in case $(\Omega, \Sigma, \mu)$ is purely atomic, $\beta_L$ is strictly coarser than $\beta_2 = \beta_1$. Since $L$ is weakly compact in $l_1$, there is a decreasing sequence of positive real numbers $\{c_n\}_{n=1}^\infty$ such that $\lim_{n \to \infty} c_n = 0$ and $L \subset \{x \in l_1\colon \sum_{k=n}^\infty |x_k| \leqslant c_n,\ n = 1, 2, \ldots\}$. Without loss of generality, we may assume that $0 < c_n \leqslant 1$ for all $n \in \mathbb{N}$. Let $M = \{x \in l_1\colon \sum_{k=n}^\infty |x_k| \leqslant \sqrt{c_n}$, $n = 1, 2, \ldots\}$. Then $M$ is weakly compact in $l_1$ and $L \subset M$. Now let $T$ be any dense, $\sigma$-compact, locally compact subset of $S$ and write $T = \bigcup_{n=1}^{\infty} T_n$, where the $T_n$ are pairwise disjoint and compact-open subsets of $S$. Define $h: T \to \mathbf{R}$ by $h = \sum_{n=1}^{\infty} \sqrt{c_n}\, \chi_{T_n}$. Then $h \in C_0(T)$ and the set

$$V = \{ g \in C(S, (l_\infty, \tau)) \colon \|g\|_{h,M} \leqslant 1 \}$$

is a $\beta_{TM}$-neighborhood of zero in $C(S, (l_\infty, \tau))$. We show that $V$ is not a $\beta_{TL}$-neighborhood of zero in $C(S, (l_\infty, \tau))$. Let $\tilde{h} \in C_0(T)$ be such that $\tilde{h}(t) \neq 0$ for all $t \in T$ and choose $m \in \mathbf{N}$ so that $|\tilde{h}(t)| < 1$ for all $t \in T_m$. Define $\tilde{g}: S \to l_\infty$ by

$$\tilde{g} = \frac{1}{\tilde{h}} \Big( \underbrace{0, 0, \ldots, 0}_{m-1}, \frac{1}{c_m}, \frac{1}{c_m}, \ldots \Big) \chi_{T_m}.$$

Then $\tilde{g} \in C(S, (l_\infty, \tau))$. Moreover, if $x \in L$, say $x = (x_1, x_2, \ldots)$ with $\sum_{k=n}^{\infty} |x_k| \leqslant c_n$ for all $n \in \mathbf{N}$, and if $t \in T_m$, then

$$\begin{aligned} |\langle x, \tilde{h}(t)\, \tilde{g}(t) \rangle| &= \left| \sum_{k=m}^{\infty} x_k \frac{1}{c_m} \right| \\ &\leqslant \frac{1}{c_m} \sum_{k=m}^{\infty} |x_k| \\ &\leqslant 1. \end{aligned}$$

So $\tilde{g} \in \{ g \in C(S, (l_\infty, \tau)) \colon \|g\|_{\tilde{h},L} \leqslant 1 \}$. But if $y = (\sqrt{c_1} - \sqrt{c_2}, \sqrt{c_2} - \sqrt{c_3}, \ldots)$, then $y \in M$ yet, for any $t \in T_m$,

$$\begin{aligned} |\langle y, h(t)\, \tilde{g}(t) \rangle| &= \frac{\sqrt{c_m}}{c_m\, |\tilde{h}(t)|} \left| \sum_{k=m}^{\infty} \sqrt{c_k} - \sqrt{c_{k+1}} \right| \\ &= \frac{1}{\sqrt{c_m}\, |\tilde{h}(t)|} \sqrt{c_m} \\ &= \frac{1}{|\tilde{h}(t)|} \\ &> 1. \end{aligned}$$

So $\tilde{g} \notin V$ and it follows that there is no basic $\beta_{TL}$-neighborhood of zero in $C(S, (l_\infty, \tau))$ which is contained in $V$. Thus, $V$ is not a $\beta_{TL}$-neighborhood of zero in $C(S, (l_\infty, \tau))$. ∎

Finally, Theorem 3.1 leads one to ask if there are examples of non-reflexive Banach spaces for which the topologies $\beta_1$ and $\beta_2$ agree even if $(\Omega, \Sigma, \mu)$ is not purely atomic. The answer to this question is not known. However, it can be shown that the $\beta_1$ and $\beta_2$ duals of $C(S, X_\tau^*)$ are the same. In fact, as we will see in the next section, both duals can be represented as $L_1(\mu, X)$.

## 4. The Strict Duals of $C(S, X_\tau^*)$

The first step in identifying the strict duals of $C(S, X_\tau^*)$ is to recognize that they are the same as the strict duals of $C_b(T, X_\tau^*)$ for any dense, $\sigma$-compact, locally compact subset $T$ of $S$. This is a consequence of the following density result.

Theorem 4.1. *$C(S, X_\tau^*)$ is dense in $(C_b(T, X_\tau^*), \beta_{TL})$ for any dense, $\sigma$-compact. locally compact subset $T$ of $S$ and any weakly compact subset $L$ of $X$.*

*Proof.* Note that $C_b(T) \otimes X^* = C(S) \otimes X^* \subset C(S, X_\tau^*) \subset C_b(T, X_\tau^*)$. By a well-known property of the general strict topology, $C_b(T) \otimes X^*$ is dense in $(C_b(T, X_\tau^*), \beta_{TL})$ [4]. Thus, $C(S, X_\tau^*)$ is dense also. ▮

Denote by $\mathrm{Bo}(S)$ the $\sigma$-algebra of Borel sets in $S$ and by $M(S)$ the space of regular measures on $\mathrm{Bo}(S)$. The next result identifies the $\beta_{TL}$ dual of $C(S, X_\tau^*)$ as a collection of $X$-valued vector measures on $\mathrm{Bo}(S)$.

Theorem 4.2. *Let $T$ be a dense, $\sigma$-compact, locally compact subset of $S$ and let $L$ be a weakly compact subset of $X$. Then*

$$\begin{aligned}(C(S, X_\tau^*), \beta_{TL})' &= (C_b(T, X_\tau^*), \beta_{TL})' \\ &= \{F\colon \mathrm{Bo}(S) \to X \mid F \text{ is a countably additive vector} \\ &\qquad \text{measure of bounded } L\text{-variation and } |F|\ (S \backslash T) = 0\}.\end{aligned}$$

*Proof.* The first equality follows from Theorem 4.1. By applying [10, Theorem 1], we see that

$$\begin{aligned}(C_b(T, X_\tau^*), \beta_{TL})' = \{F\colon \mathrm{Bo}(T) \to X \mid\ & x^* \circ F = F_{x^*} \in M(T) \\ & \text{for all } x^* \in X^* \text{ and } F \text{ is of bounded} \\ & L\text{-variation}\}.\end{aligned}$$

The requirement that $x^* \circ F \in M(T)$ for all $x^* \in X^*$ simply says that $F$ must be weakly countably additive. The Orlicz–Pettis Theorem [3, p. 22, Corollary 4] now shows that such an $F$ must be norm countably additive. The result follows by extending $F$ to $\mathrm{Bo}(S)$ and noting that for each $A \in \mathrm{Bo}(S)$, $|F|_L\ (A) = 0$ iff $|F|\ (A) = 0$ by Theorem 2.1(4). ▮

It now follows easily from the definitions of $\beta_T$ and $\beta_L$ that

$$\begin{aligned}(C(S, X_\tau^*), \beta_T)' &= (C_b(T, X_\tau^*), \beta_T)' \\ &= \{F\colon \mathrm{Bo}(S) \to X \mid F \text{ is a countably additive vector measure of bounded } L\text{-variation for some weakly compact subset } L \text{ of } X \text{ and } |F|\ (S \setminus T) = 0\}\end{aligned}$$

and

$$\begin{aligned}(C(S, X_\tau^*), \beta_L)' &= (C_b(T, X_\tau^*), \beta_L)' \\ &= \{F\colon \mathrm{Bo}(S) \to X \mid F \text{ is a countably additive vector measure of bounded } L\text{-variation and } |F|\ (S \setminus T) = 0 \text{ for each dense, } \sigma\text{-compact, locally compact subset } T \text{ of } S\}.\end{aligned}$$

Since the vector measures encountered in Theorem 4.2 are countably additive and of bounded $L$-variation, the only property which prevents them from being identified with functions in $L_1(\mu, X)$, via Lemma 2.2, is the possibility that they are not $\mu$-continuous. This difficulty disappears when the topologies under consideration are $\beta_1$ and $\beta_2$.

THEOREM 4.3. $(C(S, X_\tau^*), \beta_1)' = (C(S, X_\tau^*), \beta_2)' = L_1(\mu, X)$.

*Proof.* Since $\beta_1$ is the inductive limit of the $\beta_T$ topologies,

$$(C(S, X_\tau^*), \beta_1)' = \bigcap_T (C(S, X_\tau^*), \beta_T)'.$$

Likewise, since $\beta_2$ is the projective limit of the $\beta_L$ topologies,

$$(C(S, X_\tau^*), \beta_2)' = \bigcup_L (C(S, X_\tau^*), \beta_L)'.$$

It follows that

$$\begin{aligned}(C(S, X_\tau^*), \beta_1)' &= (C(S, X_\tau^*), \beta_2)' \\ &= \{F\colon \mathrm{Bo}(S) \to X \mid F \text{ is a countably additive vector measure of bounded } L\text{-variation for some weakly compact subset } L \text{ of } X \text{ and } |F|\ (S \setminus T) = 0 \text{ for each dense, } \sigma\text{-compact, locally compact subset } T \text{ of } S\}.\end{aligned}$$

Since a Borel set in $S$ is nowhere dense iff it has $\hat{\mu}$-measure 0, the last condition simply requires that each $F \in (C(S, X_\tau^*), \beta_1)'$ be $\hat{\mu}$-continuous. Thus, by applying Lemma 2.2 to the measure space $(S, \mathrm{Bo}(S), \hat{\mu})$, we see that $(C(S, X_\tau^*), \beta_1)' = L_1(\hat{\mu}, X)$. The natural correspondence between $L_1(\hat{\mu}, X)$ and $L_1(\mu, X)$ now yields the desired result. ∎

The actual correspondence between measures in $(C(S, X_\tau^*), \beta_1)'$ and functions in $L_1(\mu, X)$ can be described in the following way. Let $F \in (C(S, X_\tau^*), \beta_1)'$, let $\hat{f} \in L_1(\hat{\mu}, X)$ be the function guaranteed by Lemma 2.2 such that

$$F(B) = \int_B \hat{f}\, d\hat{\mu}, \qquad B \in \mathrm{Bo}(S),$$

and let $f$ be the corresponding function in $L_1(\mu, X)$. Fix $B \in \mathrm{Bo}(S)$. Then there is a unique clopen subset $\mathscr{A}$ of $S$ such that $\hat{\mu}(\mathscr{A} \triangle B) = 0$. Let $A$ be a set in $\Sigma$ corresponding to $\mathscr{A}$. Then

$$F(B) = \int_B \hat{f}\, d\hat{\mu} = \int_{\mathscr{A}} \hat{f}\, d\hat{\mu} = \int_A f\, d\mu.$$

Thus, by associating to each $B \in \mathrm{Bo}(S)$ a set $A \in \Sigma$, $F$ can be identified with the vector measure defined by $f$.

An important class of subsets of $L_1(\mu, X)$ surfaces when $L_1(\mu, X)$ is viewed as a strict dual of $C(S, X_\tau^*)$, namely, the subsets of $L_1(\mu, X)$ which are equicontinuous with respect to a strict topology on $C(S, X_\tau^*)$. In the following section, we develop characterizations for the $\beta_1$- and $\beta_2$-equicontinuous subsets of $L_1(\mu, X)$ which are closely related to the underlying measure space $(\Omega, \Sigma, \mu)$.

## 5. Strictly Equicontinuous Subsets of $L_1(\mu, X)$

Each of the characterizations presented below relies on a slight modification of a result due to Khurana [7]. It is restated here in its new form.

THEOREM 5.1. *Let $T$ be a dense, $\sigma$-compact, locally compact subset of $S$, let $L$ be a weakly compact subset of $X$, and let $K$ be a subset of $(C(S, X_\tau^*), \beta_{TL})'$. Then $K$ is $\beta_{TL}$-equicontinuous if and only if*

(1) $\sup_{F \in K} |F|_L\,(S) < \infty$ *(i.e., $K$ is uniformly bounded in L-variation) and*

(2) *for each $\varepsilon > 0$, there is a compact subset $D$ of $T$ such that $|F|_L\,(S \backslash D) < \varepsilon$ for all $F \in K$ (i.e., $K$ is uniformly TL-tight).*

*Proof.* By Theorem 4.2, $|F|\,(S \backslash T) = 0$ for each $F \in (C(S, X_\tau^*), \beta_{TL})'$. But $|F|\,(S \backslash T) = 0$ implies $|F|_L\,(S \backslash T) = 0$ by Theorem 2.1(4). Thus, $|F|_L\,(S \backslash D) = |F|_L\,(T \backslash D)$ for any $F \in (C(S, X_\tau^*), \beta_{TL})'$ and any compact subset $D$ of $T$. Moreover, $(C(S, X_\tau^*), \beta_{TL})' = (C_b(T, X_\tau^*), \beta_{TL})'$ by Theorem 4.2. The result now follows from [7, corrected Lemma 3]. ∎

COROLLARY 5.2. *Let $T$ be a dense, $\sigma$-compact, locally compact subset of $S$ and let $K$ be a subset of $(C(S, X_\tau^*), \beta_T)'$. Then $K$ is $\beta_T$-equicontinuous iff there is a weakly compact subset $L$ of $X$ such that $K$ is uniformly bounded in $L$-variation and uniformly $TL$-tight.*

*Proof.* $\beta_T$ is the projective limit of the $\beta_{TL}$ topologies on $C(S, X_\tau^*)$. ∎

By looking at sequences of clopen subsets of $S$, the property of being uniformly $TL$-tight can be viewed as a type of uniform countable additivity for the $L$-variations of the vector measures in $K$. This is especially evident in the characterization of the $\beta_1$-equicontinuous subsets of $(C(S, X_\tau^*), \beta_1)'$.

THEOREM 5.3. *Let $K$ be a subset of $(C(S, X_\tau^*), \beta_1)'$. Then the following are equivalent:*

(a) *$K$ is $\beta_1$-equicontinuous.*

(b) *For each dense, $\sigma$-compact, locally compact subset $T$ of $S$, there is a weakly compact subset $L$ of $X$ such that $K$ is both uniformly bounded in $L$-variation and uniformly $TL$-tight.*

(c) *For each sequence $\{\mathscr{A}_n\}_{n=1}^{\infty}$ of pairwise disjoint clopen subsets of $S$, there is a weakly compact subset $L$ of $X$ such that $K$ is uniformly bounded in $L$-variation and for each $\varepsilon > 0$ there is an $n_0 \in \mathbf{N}$ such that*

$$|F|_L\left(\bigcup_{i=n+1}^{m} \mathscr{A}_i\right) < \varepsilon$$

*for all $m > n \geqslant n_0$ and $F \in K$.*

(d) *For each increasing sequence $\{\mathscr{A}_n\}_{n=1}^{\infty}$ of clopen subsets of $S$, there is a weakly compact subset $L$ of $X$ such that $K$ is uniformly bounded in $L$-variation and*

$$|F|_L\left(\overline{\bigcup_{i=1}^{\infty} \mathscr{A}_i} \setminus \mathscr{A}_n\right) \xrightarrow[n \to \infty]{} 0$$

*uniformly for $F \in K$.*

*Proof.* (a) $\Leftrightarrow$ (b) This follows from Corollary 5.2 and the fact that $\beta_1$ is the inductive limit of the $\beta_T$ topologies.

(b) $\Rightarrow$ (c) Let $\{\mathscr{A}_n\}_{n=1}^{\infty}$ be a pairwise disjoint sequence of clopen subsets of $S$ and let

$$\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} \mathscr{A}_n},$$

be a clopen subset of $S$. Set

$$T=(S\backslash\mathscr{A})\cup\left(\bigcup_{n=1}^{\infty}\mathscr{A}_n\right).$$

Then $T$ is a dense, $\sigma$-compact, locally compact subset of $S$. By (b), there is a weakly compact subset $L$ of $X$ such that $K$ is uniformly bounded in $L$-variation and uniformly $TL$-tight. Let $\varepsilon>0$ and choose a compact subset $D$ of $T$ so that $|F|_L(S\backslash D)<\varepsilon$ for all $F\in K$. Since $D$ is compact and covered by the open sets $(S\backslash\mathscr{A})$, $\mathscr{A}_1$, $\mathscr{A}_2$, ..., there is an $n_0\in\mathbf{N}$ such that

$$D\subset(S\backslash\mathscr{A})\cup\left(\bigcup_{i=1}^{n_0}\mathscr{A}_i\right).$$

Thus, for $F\in K$ and $m>n\geqslant n_0$,

$$|F|_L\left(\bigcup_{i=n+1}^{m}\mathscr{A}_i\right)\leqslant|F|_L(S\backslash D)<\varepsilon.$$

(c) $\Rightarrow$ (d) Straightforward.

(d) $\Rightarrow$ (b) Let $T$ be a dense, $\sigma$-compact, locally compact subset of $S$. Write $T=\bigcup_{n=1}^{\infty}T_n$, where each $T_n$ is compact-open and $T_1\subset T_2\subset\cdots$. Then $\{T_n\}_{n=1}^{\infty}$ is an increasing sequence of clopen subsets of $S$ with

$$S=\overline{\bigcup_{n=1}^{\infty}T_n}.$$

By (d) it follows that there is a weakly compact subset $L$ of $X$ such that $K$ is uniformly bounded in $L$-variation and $|F|_L(S\backslash T_n)\to 0$ as $n\to\infty$ uniformly for $F\in K$. Let $\varepsilon>0$ and choose $n_0\in\mathbf{N}$ so that $|F|_L(S\backslash T_n)<\varepsilon$ for all $n\geqslant n_0$ and $F\in K$. Take $D=T_{n_0}$ to obtain $|F|_L(S\backslash D)<\varepsilon$ for all $F\in K$. ▌

Condition (c) of Theorem 5.3 can be simplified considerably in the characterization of the $\beta_L$-equicontinuous subsets of $(C(S, X_\tau^*), \beta_L)'$ since, in this case, a single weakly compact subset of $X$ works for each sequence of pairwise disjoint clopen subsets of $S$.

THEOREM 5.4. *Let $L$ be a weakly compact subset of $X$ and let $K$ be a subset of $(C(S, X_\tau^*), \beta_L)'$. Then the following are equivalent:*

(a) *$K$ is $\beta_L$-equicontinuous.*

(b) *$K$ is uniformly bounded in $L$-variation and uniformly $TL$-tight for each dense, $\sigma$-compact, locally compact subset $T$ of $S$.*

(c) *$K$ is uniformly bounded in $L$-variation and, for each sequence $\{\mathscr{A}_n\}_{n=1}^{\infty}$ of pairwise disjoint clopen subsets of $S$, $|F|_L(\mathscr{A}_n) \to 0$ as $n \to \infty$ uniformly for $F \in K$.*

(d) *$K$ is uniformly bounded in $L$-variation and, for each increasing sequence $\{\mathscr{A}_n\}_{n=1}^{\infty}$ of clopen subsets of $S$,*

$$|F|_L\left(\overline{\bigcup_{i=1}^{\infty} \mathscr{A}_i} \setminus \mathscr{A}_n\right) \xrightarrow[n \to \infty]{} 0$$

*uniformly for $F \in K$.*

*Proof.* Similar to the proof of Theorem 5.3. ∎

The final result of this section is an immediate consequence of the fact that $\beta_2$ is the projective limit of the $\beta_L$ topologies. Combined with Theorem 5.4, it yields a characterization for the $\beta_2$-equicontinuous subsets of $L_1(\mu, X)$.

THEOREM 5.5. *Let $K$ be a subset of $(C(S, X_\tau^*), \beta_2)'$. Then $K$ is $\beta_2$-equicontinuous iff it is $\beta_L$-equicontinuous for some weakly compact subset $L$ of $X$.*

When referring to subsets of $L_1(\mu, X)$, it is convenient to make certain modifications to the characterizations presented above. Note first that for a function $f \in L_1(\mu, X)$ and a weakly compact subset $L$ of $X$, the $L$-variation of $f$ over a set $A \in \Sigma$ is given by

$$|f|_L(A) = \sup\left\{ \sum_{i=1}^{m} \left| \left\langle \int_{A_i} f\, d\mu, x_i^* \right\rangle \right| : \{A_i\}_{i=1}^{m} \text{ is a } \Sigma\text{-partition of } A,\ \{x_i^*\}_{i=1}^{m} \subset L^o \right\}.$$

Now by applying Theorem 5.3(d), a subset $K$ of $L_1(\mu, X)$ is $\beta_1$-equicontinuous if and only if for each increasing sequence $\{A_n\}_{n=1}^{\infty}$ of sets in $\Sigma$, there is a weakly compact subset $L$ of $X$ such that

(1) $\sup_{f \in K} |f|_L(\Omega) < \infty$ (i.e., $K$ is uniformly bounded in $L$-variation) and

(2) $|f|_L((\bigcup_{i=1}^{\infty} A_i) \setminus A_n) \to 0$ as $n \to \infty$ uniformly for $f \in K$.

A subset $K$ of $L_1(\mu, X)$ which satisfies these conditions, or the equivalent conditions arising from Theorem 5.3(c), will be said to be *strictly equicontinuous*. Likewise, using Theorems 5.4(d) and 5.5, a subset $K$ of $L_1(\mu, X)$ is $\beta_2$-equicontinuous if and only if there is a weakly compact subset $L$ of $X$ such that

(1′) $K$ is uniformly bounded in $L$-variation and

(2′) for each increasing sequence $\{A_n\}_{n=1}^{\infty}$ of sets in $\Sigma$, $|f|_L((\bigcup_{i=1}^{\infty} A_i)\backslash A_n) \to 0$ as $n \to \infty$ uniformly for $f \in K$.

A subset $K$ of $L_1(\mu, X)$ which satisfies these conditions, or the equivalent conditions arising from Theorem 5.4(c), will be said to be *strongly strictly equicontinuous*. The distinguishing characteristic between the strictly equicontinuous and strongly strictly equicontinuous subsets of $L_1(\mu, X)$ is the fact that the $L$ for the former depends on the sequence $\{A_n\}_{n=1}^{\infty}$ while the $L$ for the latter is the same for any such sequence. At present, it is not known whether or not strictly equicontinuous subsets of $L_1(\mu, X)$ are always strongly strictly equicontinuous. Naturally, an answer to this question would also answer the question as to whether or not the topologies $\beta_1$ and $\beta_2$ always coincide.

## References


1. R. Bourgin, Geometric aspects of convex sets with the Radon-Nikodým property, *in* "Lecture Notes in Math.," Vol. 993, Springer-Verlag, Berlin, 1983.
2. R. C. Buck, Bounded continuous functions on a locally compact space, *Michigan Math. J.* **5** (1958), 95–104.
3. J. Diestel and J. J. Uhl, Vector measures, *in* "Mathematical Surveys," Vol. 15, Amer. Math. Soc., Providence, RI, 1977.
4. R. Fontenot, Strict topologies for vector-valued functions, *Canad. J. Math.* **26** (1974), 841–853; Addendum **27** (1975), 1183–1184.
5. A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, *Mem. Amer. Math. Soc.* **16** (1955).
6. M. Gruenwald, "Strictly Equicontinuous Subsets of the Space of Bochner Integrable Functions," Dissertation, Northern Illinois University, 1989.
7. S. Khurana, Strict topology on paracompact locally compact spaces, *Canad. J. Math.* **29** (1977), 216–219; Addendum **30** (1978), 671.
8. D. Sentilles, Stone space representation of vector functions and operators on $L^1$, *in* "Proceedings Conf. Measure Theory, Oberwolfach," Lecture Notes in Math., Vol. 794, pp. 470–473, Springer-Verlag, New York, 1980.
9. G. Schlüchtermann and R. Wheeler, On strongly WCG Banach spaces, *Math. Z.* **199** (1988), 387–398.
10. J. Wells, Bounded continuous vector-valued functions on a locally compact space, *Michigan Math. J.* **12** (1965), 119–126.
11. R. Wheeler, Seminar notes, 1983.